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Heat transfer from a stretching surface

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1. INTRODUCTION

THE HEAT transfer from a stretching surface is of interest in polymer extrusion processes where the object, after passing through a die, enters the fluid for cooling below a certain temperature. The rate at which such objects are cooled has an important bearing on the properties of the final product. In the cooling fluids the momentum boundary layer for linear stretching of sheet $U \propto x$ was first studied by Crane [1], whereas power law stretching $U \propto x^m$ was initially described by Afzal and Varshney [2].

Heat transfer from a linearly stretched surface $U \propto x$ based on the above work [1] has attracted the attention of several workers. The case of constant wall temperature has already been the subject of study [1, 3]. Similarly, for a non-uniform wall temperature closed form solution in terms of special functions has also been reported [5]. The case of uniform sheet velocity (zero stretching) is also well documented [7, 8].

The present work deals with heat transfer from an arbitrarily stretching surface $U \propto x^m$ for investigating the effects of non-uniform surface temperature. Several closed form solutions for specific values of m including their numerical solutions are presented in this technical note.

2. EQUATIONS

Let a polymer sheet emerging out of a slit at origin ($x = 0$) be moving with non-uniform velocity $U(x)$ in an ambient fluid at rest. The coordinate systems shown in Fig. 1, where coordinate x is the direction of motion of the sheet and y is the coordinate normal to it. The u and v are velocity components in the x and y directions, respectively. Further, ν is the molecular kinematic viscosity and σ the Prandtl number of the fluid. The boundary layer equations of mass, momentum and energy for two-dimensional constant pressure flow in usual notations are as follows:

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = \nu u_{yy} \tag{2}$$

$$uT_x + vT_y = \sigma^{-1}\nu T_{yy}. \tag{3}$$

The boundary conditions for the flow induced by stretching sheet (issuing from the slit $x = 0$) moving with non-uniform surface speed $U(x)$ in quiescent environment are:

$$y = 0, \quad u = U(x), \quad v = 0, \quad T = T_w(x) \tag{4}$$

$$y/\delta \to \infty, \quad u \to 0, \quad T \to T_\infty. \tag{5}$$

Introducing the similarity variables

$$\psi = \nu\sqrt{(2\xi)}f(\eta), \quad \eta = \frac{Uy}{\nu\sqrt{(2\xi)}}, \quad \xi = \frac{Ux}{\nu(m+1)}$$

$$T = T_\infty + (T_w - T_\infty)\theta(\eta)$$

$$U = U_0 x^m, \quad T_w = T_\infty + Cx^n, \quad \beta = \frac{2m}{1+m} \tag{6}$$

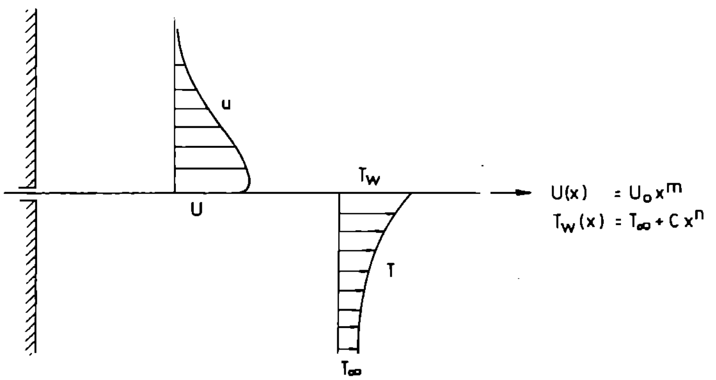

FIG. 1. Coordinate system for the flow induced by a polymer sheet moving with non-uniform surface speed in an ambient fluid at rest.

the momentum boundary layer equations reduce to

$$f'''+ff''-\beta f'^2 = 0$$

$$f(0) = 0,\quad f'(0) = 1,\quad f'(\infty) = 0 \qquad (7)$$

and thermal boundary layer equations to

$$\sigma^{-1}\theta''+f\theta'-n(2-\beta)f'\theta = 0$$

$$\theta(0) = 1,\quad \theta(\infty) = 0. \qquad (8)$$

Equations (7) describing the flow on a stretching sheet were first proposed by Afzal and Varshney [2] and the solutions for $-2 < \beta < 2$ were also reported. The later investigations [10-13] connected with these equations (7) have not cited the original paper [2], while the bulk of earlier results [2] have been entirely reproduced/adopted in these papers. On the other hand Aziz and Na [14] in their book referred to equations (7) with reference to the original work [2].

3. MOMENTUM BOUNDARY LAYER

When β is large the solution can be obtained by the method of matched asymptotic expansions [9]. In the outer layer (η fixed $\beta \to \infty$) equations (7) show that to order $1/\beta$ the function f is constant, which fails to satisfy the no slip condition. In the inner variables [9]

$$F(Z) = f(\eta)/\varepsilon,\quad Z = \eta/\varepsilon,\quad \varepsilon = (6/\beta)^{1/2} \qquad (9)$$

the boundary layer equations (7) become

$$F'''-6F'^{2} = -\varepsilon^2 FF''$$

$$F(0) = 0,\quad F'(0) = 1.$$

Using the inner expansion

$$F = F_0+\varepsilon^2 F_1+\cdots \qquad (10)$$

the solution to equations for F_0 and F_1 which matches with the outer solution is

$$F_0 = \frac{Z}{1+Z}$$

$$F_1 = \frac{1}{15}\left[-3\ln(1+Z)+4-\frac{5}{1+Z}+\frac{1}{(1+Z)^2}\right]. \qquad (11)$$

The velocity gradient at the wall is given by

$$f''(0) = -\left(\frac{2\beta}{3}\right)^{1/2}\left[1+\frac{1}{5\beta}+\frac{18}{125\beta^2}+\cdots\right]. \qquad (12)$$

4. THERMAL BOUNDARY LAYER

For $\beta = 1$ closed form solution of the thermal boundary layer equations (8) has been reported [5]. The solutions for other specific values of β are described here.

(a) $\beta = -1$: based on the solution of momentum boundary layer equations (7)

$$f(\eta) = (2X)^{1/2},\quad X = \tanh^2(\eta/\sqrt{2}) \qquad (13)$$

the thermal boundary layer equations (8) after some manipulation can be expressed as

$$X(1-X)h_{xx}+[c-(a+b+1)X]h_x-abh = 0 \qquad (14)$$

$$h(0) = 1,\quad h(1) = 0 \qquad (15)$$

where $h(X) = \theta(\eta)$ and constants a, b and c are given by

$$a+b = (1-2\sigma)/2,\quad ab = 3n\sigma/2,\quad c = 1/2. \qquad (16)$$

Equation (14) is the well-known hypergeometric equation [15] and its solution satisfying boundary conditions (15) is

$$h = F(a,b,\tfrac{1}{2},X)+D(2X)^{1/2}F(\tfrac{1}{2}+a,\tfrac{1}{2}+b,\tfrac{3}{2},X) \qquad (17)$$

$$D = -\sqrt{2}\Gamma(1-a)\Gamma(1-b)/[\Gamma(\tfrac{1}{2}-a)\Gamma(\tfrac{1}{2}-b)] \qquad (18)$$

where $F(a, b, c, X)$ is the hypergeometric function and $\Gamma(z)$ the complete gamma function [15]. Further constants a and b from (16) are given by

$$(a,b) = [1-2\sigma\mp\{(1-2\sigma)^2-24n\sigma\}^{1/2}]/4. \qquad (19)$$

The temperature gradient on the sheet is

$$\theta'(0) = D. \qquad (20)$$

Solution (17) for specific values of n can be expressed into a simple function. If $n = -1$ it becomes

$$\theta = (1-X)^{\sigma} = \operatorname{sech}^{2\sigma}(\eta/\sqrt{2}) \qquad (21)$$

and if $n = 0$ then

$$\theta = 1-B_x(\tfrac{1}{2},\sigma)/B_1(\tfrac{1}{2},\sigma)$$

$$= -\sqrt{\left(\frac{2}{\pi}\right)}\frac{\Gamma(\frac{1}{2}+\sigma)}{\Gamma(\sigma)} \qquad (22)$$

where $B_x(n,m)$ is the incomplete beta function defined by

$$B_x(m,n) = \int_0^x z^{m-1}(1-z)^{n-1}\,dz. \qquad (23)$$

If $\sigma = 1$ solution (22) is further reduced to

$$\theta = 1-\sqrt{X},\quad \theta'(0) = -1/\sqrt{2}. \qquad (24)$$

(b) For $\sigma = 1$, $n = \beta/(2-\beta)$ the closed form solution of energy equation (8) is

$$\theta' = f \qquad (25)$$

which shows that classical Reynolds analogy between momentum and heat transfer is also valid for flow on stretching sheet having non-uniform surface temperatures.

(c) If $n(2-\beta) = -1$ the closed form solution of (8) is

$$\theta(\eta) = \exp\left(-\sigma\int_0^{\eta} f\,d\eta\right),\quad \theta'(0) = 0. \qquad (26)$$

(d) For general values of β and n solutions of energy equation (8), for limiting Prandtl numbers have been obtained by the method of matched asymptotic expansions. The algebra is complicated and for brevity, only final results are given here. For $\sigma \to \infty$ thermal boundary layer thickness is of the order of $\sigma^{-1/2}$ and heat transfer rate is given by

$$\theta'(0) = -\sqrt{(2\sigma)}\alpha+f''(0)[(2K+1)\alpha^2-2K\alpha]/3+\cdots$$

where

$$\alpha = \Gamma\left(\frac{K}{2}+1\right)\Big/\Gamma\left(\frac{1+K}{2}\right),\quad K = n(2-\beta). \qquad (27)$$

For $\sigma \to 0$, thermal boundary layer thickness is of the order of σ and the heat transfer rate is

$$\theta'(0) = -(1+K)f(\infty)\sigma+(1+K)^2[KJ_2-f(\infty)J_1]\sigma^2+\cdots$$

$$J_m = \int_0^{\infty}[f(\eta)-f(\infty)]^m\,d\eta. \qquad (28)$$

(e) For $\beta \to \infty$ the analysis by matched asymptotic expansions shows that the solution exists only for $n < 0$ when the heat transfer rate is given by

$$\theta'(0) = \frac{\sigma}{1+a}\sqrt{(6\beta)}\left[n-\frac{1+2n}{\beta}+0(\beta^{-2})\right],\quad n < 0$$

$$2a = (1-24n\sigma)^{1/2}-1. \qquad (29)$$

4. RESULTS AND DISCUSSION

The asymptotic solution (12) of momentum boundary layer for $\beta \to \infty$ is displayed in Fig. 2. The predictions are very good even for $\beta = 1$ where it predicts -1.079 against exact value of -1. The series (12) can further be improved through conventional Euler transformation by recasting it

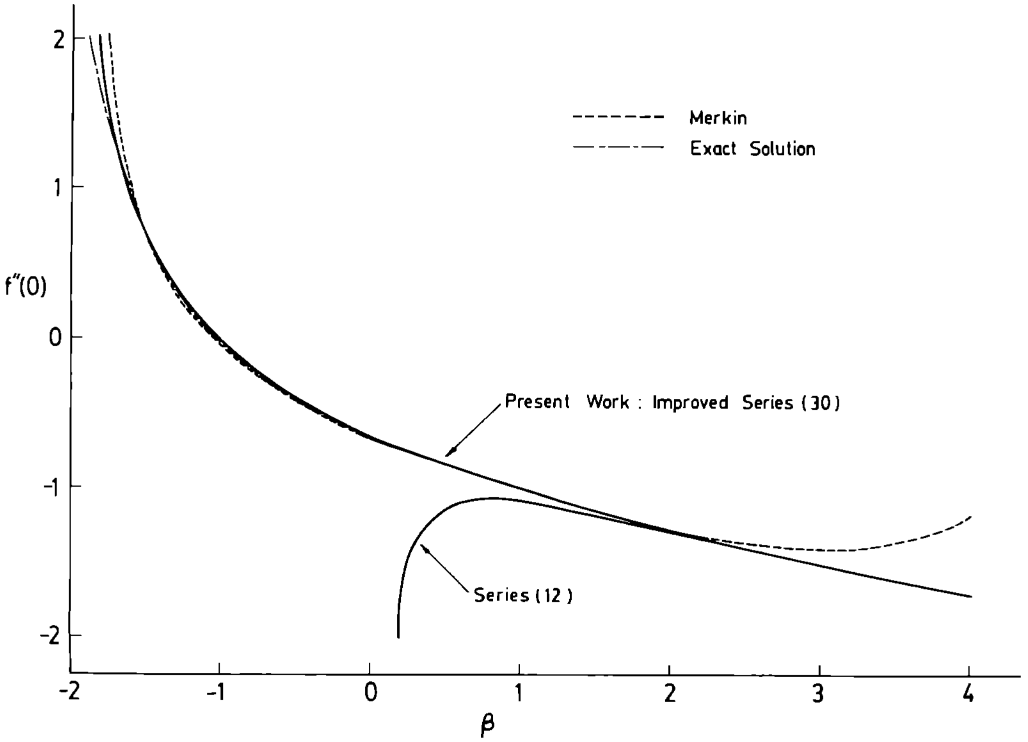

FIG. 2. A comparison of velocity gradient $f''(0)$ at the stretching sheet from asymptotic solution at large β with available solutions: ——, improved asymptotic series (30); – – –, Merkin's [12] series solution; —·—, exact results; ——, three term solution (12) for $\beta \to \infty$.

in terms of variable $E = 1/(\beta+3)$ and noting the fact that as $\beta \to -2$, $(2+\beta)^{3/4}f''(0)$ approaches a constant [11] to get

$$(2+\beta)^{3/4}f''(0) = -\left(\frac{4}{9E}\right)^{1/4}\left[\frac{1}{E}-\frac{41}{40}+\frac{81}{4000}E+\cdots\right]. \quad (30)$$

As $E \to 1(\beta \to -2)$, the right-hand side of (30) predicts 0.694 whereas the exact result [11] is 0.743. Relation (30) for $\beta = 1$, 0 and -1 predicts -1.0134, -0.6496 and -0.049 whereas correspondingly exact results are represented by -1.0, -0.6275 and 0.0. The series solution of Merkin [12] as displayed in Fig. 3 holds good for $-1.5 < \beta < 2$ and fails when β becomes larger than 2 or approaches -2. In conclusion, relation (30) predicts $f''(0)$ in the entire domain of β ranging from infinity to -2.

Numerical solutions to thermal boundary layer equations (8) have been obtained for various values of β and n pertaining to two values of Prandtl number $\sigma = 0.72$ (air) and 7 (water). The wall temperature gradient $\theta'(0)$ is displayed in Fig. 1 for $\sigma = 0.72$. For $n > 0$, or more precisely

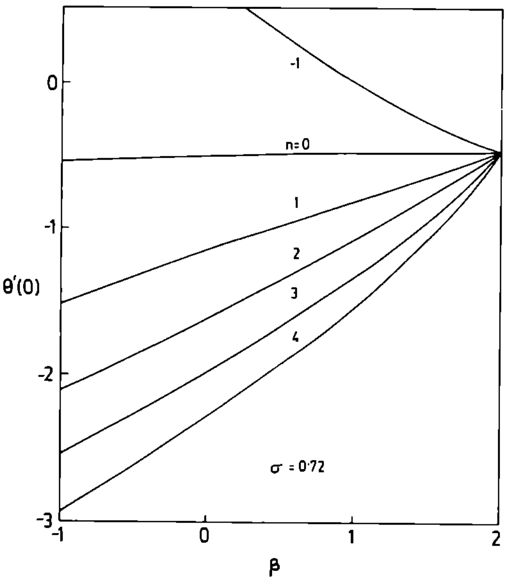

FIG. 3. Heat transfer rate on a non-isothermal stretching sheet for $\sigma = 0.72$.

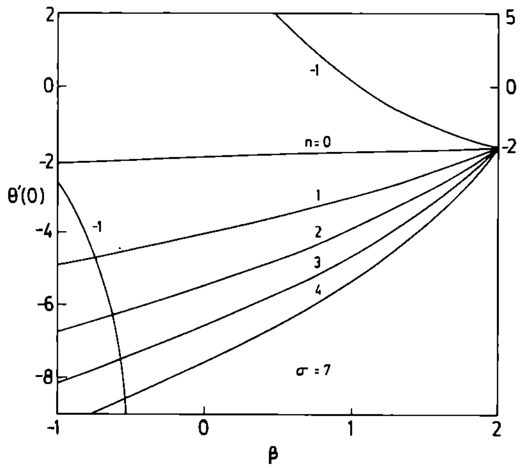

FIG. 4. Heat transfer rate on a non-isothermal stretching sheet for $\sigma = 7.0$.

$n(2-\beta) > 0$ the temperature gradient $\theta'(0)$ is negative when the heat flows from a surface to the ambient fluid. For a fixed value of n the magnitude of heat transfer rate decreases as β increases. For a fixed β the magnitude of heat transfer increases as n increases. For $\sigma = 7$ (water), the numerical solutions are displayed in Fig. 4. For $n = -1$ the heat transfer rate changes sign through an infinite discontinuity, for $\sigma = 7$ it occurs around $\beta = 0$. This behaviour is also shown by closed form solutions, but such a behaviour is physically unrealistic and corresponds to breakdown of boundary layer theory. The closed form solutions presented in Section 3 agree with the derived numerical solutions.

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